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RADIATIVE CONTRIBUTIONS TO THE EFFECTIVE ACTION OF SELF-INTERACTING SCALAR FIELD ON A MANIFOLD WITH BOUNDARY

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Abstract. The effect of quantum corrections to a conformally invariant field theory for a self-interacting scalar field on a curved manifold with boundary is considered. The analysis is most easily performed in a space of constant curvature the boundary of which is characterised by constant extrinsic curvature. An extension of the spherical formulation in the presence of a boundary is attained through use of the method of images. Contrary to the consolidated vanishing effect in maximally symmetric space-times the contribution of the massless " tadpole" diagram no longer vanishes in dimensional regularisation. As a result, conformal invariance is broken due to boundary-related vacuum contributions. The evaluation of one-loop contributions to the two-point function suggests an extension, in the presence of matter couplings, of the simultaneous volume and boundary renormalisation in the effective action.

I. Introduction

The investigation of elliptic operators on Riemannian manifolds with boundary is a pivotal aspect in Euclidean Quantum Gravity. The eigenvalue problem for operators of Laplace type, in particular, arises naturally in the context of the semiclassical approximation to the wave function of the Universe. Specifically, instanton related considerations in quantum and inflationary cosmology lend particular importance to the issue of radiative contributions to a semiclassical tunneling geometry of constant curvature, effected by quantised matter. In the case of a quantised scalar field coupled to the background geometry such an issue relates naturally to the eigenvalue problem of the Laplace operator formulated on a bounded segment of the de Sitter four-dimensional sphere. For this spatially symmetric situation it is possible to relate the eigenvalue problem on the bounded region C of the spherical cap to one on its covering manifold which constitutes the entire four sphere. The method of images can accomplish this task provided that either Dirichlet or Neumann (but not Robin) conditions are specified on the boundary for the scalar field [1]. The latter provide also a sufficient condition for self adjointness of the Laplace operator.

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As a consequence of the investigation of elliptic operators heat kernel and functional methods have been invoked for the evaluation of the one-loop semiclassical approximation to the functional integral for quantum gravity, that is to the formal sum over a specified set of geometries meeting the boundary conditions [2], [3]. These methods have been extended in the presence of matter couplings [4], [5], [6], [7]. Although use of renormalisation group techniques have yielded the improved effective action past one-loop order [7] no attempt has hitherto been made for the evaluation of higher loop-order contributions to semiclassical tunnelling geometries past that level. Such calculations would necessarily rely on diagramatic techniques on a manifold with boundary. The relevant approach would be predicated on the concept of the relativistic propagator and would be distinct from diagramatic techniques related to heat kernel asymptotic expansions hitherto proposed [8], [9]. Fundamental in this calculational context is the evaluation of the contribution which the boundary of the manifold has to the propagator of the relevant quantised matter field coupled to the manifold's semiclassical background geometry. In the physically important case of a n-dimensional bounded spherical cap C_n the method of images can be invoked as the computational technique for the matter propagator. In relating the relevant eigenvalue problem on C_n to one on its covering manifold the method of images essentially relates propagation on C_n to propagation on the entire S_n. This is the case because the propagator is necessarily the Green function associated with the bounded elliptic operator on C_n. Specifically, in the case of massless propagation on C_n for a scalar field conformally coupled to the background geometry both the fundamental part of the Green function and the additional to it part arising from boundary conditions specified on ∂C_n admit an expansion in terms of the eigenfunctions of the Laplace operator defined on S_n provided that the method of images is used. In effect, a consistent use of the method of images dispenses with the rather intractable expansions involving spherical harmonics of fractional degrees which naturally arise as eigenfunctions of the bounded Laplace operator on C_n.

The present work addresses the issue which the presence of a $(n-1)$-dimensional boundary of constant extrinsic curvature on a Riemannian n-manifold of constant curvature raises for the dynamical behaviour of a conformal scalar field defined on that manifold. The presence of such boundaries no longer allows for the established spherical formulation of conformal scalar theories as it destroys the SO(n+1) (de Sitter) invariance on which the latter are predicated. It will be shown that in relating the eigenvalue problem for the bounded Laplace operator on the n-cap to that on the entire n-sphere the method of images allows for a simple solution to the Green equation with a direct physical interpretation. As a precursor to a renormalisation program a conformal scalar field supplemented with a self-interacting ϕ^4 coupling will be considered in the $n=4$ case. It will be shown that the vanishing effect which the technique of dimensional regularisation is known to have on the massless "tadpole" diagram on maximally symmetric manifolds is no longer sustainable in the present case of the stated bounded manifold, a result which suggests general significance for that vanishing effect only on maximally symmetric manifolds. The boundary effect on the "tadpole" diagram is finite as a consequence of which to first order in the self-coupling the scalar vacuum effects do not result in infinite contributions to the semi-classical boundary part of the bare action. At higher

orders, nevertheless, more complicated diagramatic structures entail the potential for generation of infinite redefinitions. It will be shown that the relevant loop integrals have the potential for the simultaneous redefinition of the volume-related terms in the bare action and of those terms in the latter which the presence of the boundary ∂C_n necessitates at the semi-classical level as a condition for the validity of the Einstein equations on C_n. This allows for the generalisation to arbitrary loop-orders of the simultaneous renormalisation of boundary and volume ultra-violet divergences attained at one-loop level through use of heat kernel techniques [10]. Moreover, the volume contribution of the "tadpole" to the effective action will be shown to have the potential for generation of conformally non-invariant counterterms at higher orders.

II. The method of images and propagation on the bounded manifold

The spherical formulation of the dynamical behaviour of a conformal scalar field ϕ admitting a classical action which remains invariant under the conformal rescaling

$$g_{\mu\nu} \longrightarrow \Omega^2(x) g_{\mu\nu}, \ \ \phi \longrightarrow \Omega^{1-\frac{1}{2}n}\phi \equiv \Phi \tag{1}$$

in a general n-dimensional Lorentzian space-time, is attained on a Riemannian manifold by specifying its background geometry to be that of positive constant curvature r embedded in a $(n+1)$-dimensional space. The dynamical behaviour of self-interacting scalar models has been studied in this context for $n = 4$ and $n = 3$ [11], [12], [13] by exploiting the coincidence between the classical action of the theory specified on the n-sphere S_n and the classical action obtained by conformally mapping the theory specified on the n-dimensional Euclidean space, onto S_n. The spherical n-cap C_n considered as a manifold of constant curvature embedded in a $(n+1)$-dimensional Euclidean space and bounded by a $(n-1)$-sphere of positive extrinsic curvature k (diverging normals) is, in the same respect, conformal to the interior of the associated $(n-1)$-sphere which constitutes the n-disk embedded in the same $(n+1)$-space. The choice of transformation (1) which maps Euclidean space-time onto a de Sitter sphere and thereby the n-disk onto the n-cap is, for many purposes, arbitrary. The usual technique of stereographic projection will be invoked for the purposes of the ensuing analysis. In conformity with the results in [11], a massless scalar field ϕ on the n-dimensional disk with the Dirichlet condition $\phi = 0$ specified on the disk's boundary is mapped on the n-cap into the spherical form

$$\Phi = \kappa^{1-\frac{n}{2}}\phi \tag{2}$$

with the multiplicative constant

$$\frac{\partial(\eta)}{\partial(x)} = \kappa^{n+1} \tag{3}$$

relating to the Jacobian of the conformal transformation $x \longrightarrow \eta$, x and η being $n+1$ vectors. The same condition $\Phi = 0$ on the cap's $(n-1)$-dimensional spherical boundary emerges naturally through this transformation. In effect, the kinetic part of the scalar action on the cap becomes

$$\frac{1}{2}\int d^n\eta\Phi M_c\Phi = \frac{1}{2}\int d^n x\phi\partial^2\phi \tag{4}$$

where integration is understood over the cap and the disk volume respectively. M_c is the relevant bounded spherical Laplace operator conformally related to the d'Alembertian ∂^2, its exact form remaining the same as that of the unbounded Laplace operator M defined on the entire S_n

$$M_c = D^2 - \frac{n(n-2)}{4R^2};\ D_a = (\delta_{ab} - \frac{\eta_a\eta_b}{R^2})\frac{\partial}{\partial\eta_a} \tag{5}$$

but its domain being different due to the presence of the boundary. In effect, the spherical harmonics Y_α^N of integral degrees of homogeneity N which in $n+1$ dimensions form a complete set of eigenfunctions for the corresponding Laplacian defined on S_n are no longer eigenfunctions of M_c. The non trivial boundary conditions alter the spectrum of eigenvalues thereby directly affecting the corresponding degrees N which become, in turn, fractional as a condition for orthonormality. It has been shown, in fact, that cap spherical harmonics do not actually form a complete set [14].

It is evident at this point that the presence of a boundary on a manifold of constant curvature is incompatible with a direct application of the spherical formulation. Moreover, the stated lack of completeness and the emergence of fractional values for the degrees N which are physically associated with angular momenta flowing through the relevant propagators would tend to obscure a direct physical interpretation of any perturbative calculation. Such complications are the direct result of the smaller symmetry group which is retained for the Riemannian bounded manifold from the original SO(n+1) invariance. It would, for that matter, be desirable to relate the eigenvalue problem for the bounded Laplace operator M_c defined on C_n to that of the unbounded Laplacian M on S_n, the covering manifold of C_n. The method of images is the simplest expedient to this end and can be best applied on the embedded n-dimensional disk which is conformal to C_n rather than on C_n itself. Specifically, for $n > 2$ the Green's function to the Euclidean n-dimensional ∂^2 is $|x-x'|^{2-n}$. The presence of the disk's boundary on which the condition $\phi = 0$ has been specified generates an additional term which remains finite at the limit $x \to x'$, for any x' in the volume of the disk. In effect, the massless scalar propagator on the n-disk is

$$D^{(n)}(x,x') = \frac{1}{|x-x'|^{n-2}} - \frac{1}{|\frac{|x'|}{r}x - \frac{r}{|x'|}x'|^{n-2}} \tag{6}$$

with r being the radius of the $(n-1)$-sphere which constitutes the disk's boundary. This propagator is in conformity with the stated Dirichlet condition and reduces to the flat-space propagator for the massless scalar field at the $r \to \infty$ limit.

The propagator on C_n can now be obtained by exploiting the conformal relation between the latter and the n-disk. Crucial to the general significance of this approach is the aforementioned equivalence between the action for the scalar field conformally coupled to the background geometry of C_n and that obtained through a conformal transformation

of the action for a massless scalar field specified on the n-disk [12]. The stereographic projection applied with reference to the north pole on S_n maps the centre of the n-disk onto the north pole's counterdiametric pole, that is, on the pole of C_n. In effect, r is mapped onto the geodesic distance a_B between the cap's pole and any point on the boundary S_{n-1} of the cap. As a result, $D^{(n)}(x, x')$ is mapped onto

$$D_c^{(n)}(\eta, \eta') = \frac{1}{|\eta - \eta'|^{n-2}} - \frac{1}{|\frac{a_{\eta'}}{a_B}\eta - \frac{a_B}{a_{\eta'}}\eta'|^{n-2}} \tag{7}$$

with $a_{\eta'}$ being the geodesic distance between the cap's pole and η'. $D_c^{(n)}$ is the desired propagator for a conformal scalar field on C_n. It possesses the same structure as that of its conformal counterpart and satisfies

$$M_c D_c^{(n)}(\eta, \eta') = \delta^{(n)}(\eta - \eta') \tag{8}$$

with the condition $D_c^{(n)}(\eta, \eta') = 0$ when $a_{\eta'} = a_B$ which guarrantees the absence of propagation on the boundary hypersurface. The fundamental part $|\eta - \eta'|^{n-2}$ of this Green function is the usual Euclidean de Sitter space propagator $D_s^{(n)}(\eta, \eta') = 0$. The additional to it boundary term expresses the contribution due to reflection off the boundary hypersurface and remains finite at the coincidence limit $\eta \to \eta'$ in the volume of C_n. It can be seen, however, to develop the same divergence at the coincidence limit on ∂C_n as that of the fundamental part. This divergence is a direct consequence of the demand for vanishing propagation on the boundary and will be shown to be responsible for the presence of Φ-related boundary terms in the effective action in spite of the Dirichlet condition $\Phi = 0$ on ∂C_n. It is also worth noting that $D_c^{(n)}$ reduces to $D_s^{(n)}$ for $a_{\eta'} = 0$, that is, for any propagation to the pole of C_n. The physical significance of this effect, which is also characteristic of massless scalar propagation on the n-disk, stems from the invariance group of C_n and apparently refers to the unique cancellation of all contributions from the boundary to pole-directed propagation.

The fundamental part of $D_c^{(n)}(\eta, \eta')$ is the elementary Haddamard function admitting the expansion

$$|\eta - \eta'|^{2-n} = \sum_{N=0}^{\infty} \sum_{\alpha=0}^{N} \frac{1}{\lambda_N} Y_\alpha^N(\eta) Y_\alpha^N(\eta') \tag{9}$$

in terms of the complete set of spherical harmonics in $n+1$ dimensions [11]

$$\int d^n\eta Y_\alpha^N(\eta) Y_{\alpha'}^{N'}(\eta) = \delta_{NN'}^{(n)} \delta_{\alpha\alpha'}^{(n)} \tag{10}$$

$$\sum_{N=0}^{\infty} \sum_{\alpha=0}^{N} Y_\alpha^N(\eta) Y_\alpha^N(\eta') = \delta^{(n)}(\eta - \eta') \tag{11}$$

which are eigenfunctions of the Laplacian M on the embedded S_n of radius a, the covering manifold of C_n

$$MY_\alpha^N(\eta) = \lambda_N Y_\alpha^N(\eta) \tag{12}$$

$$\lambda_N = -\frac{(N+\frac{n}{2}-1)(N+\frac{n}{2})}{a^2} \tag{13}$$

At the same time, the expression $|\frac{a_{\eta'}}{a_B}\eta - \frac{a_B}{a_{\eta'}}\eta'|$ in the boundary part of the propagator in (7) is the geodesic distance between the associated two points $\frac{a_{\eta'}}{a_B}\eta$ and $\frac{a_B}{a_{\eta'}}\eta'$. In fact, $\frac{a_B}{a_{\eta'}} > 1$ indicates that this geodesic distance is relevant to the entire S_n rather than merely to C_n. In conformity with the method of images this allows for the interpretation of the boundary contributions to massless scalar propagation between η and η' on C_n as arising from propagation on S_n. However, the condition of a lower, non-vanishing limit is enforced on this propagation by the coincidence limit $\eta \to \eta'$ on C_n. The propagation on S_n representing the boundary contributions does not occur for geodesic separations smaller than $|\frac{a_{\eta'}}{a_B}\eta' - \frac{a_B}{a_{\eta'}}\eta'|$. In effect, the boundary part in (7) being itself an elementary Haddamard function admits the same expansion

$$|\frac{a_{\eta'}}{a_B}\eta - \frac{a_B}{a_{\eta'}}\eta'|^{2-n} = \sum_{N=0}^{N_0}\sum_{\alpha=0}^{N}\frac{1}{\lambda_N}Y_\alpha^N(\frac{a_{\eta'}}{a_B}\eta)Y_\alpha^N(\frac{a_B}{a_{\eta'}}\eta') \tag{14}$$

with the condition that now only a finite number of terms in the complete set $Y_\alpha^N(\eta)$ is relevant as a result of the cut-off separation. The integer degree N in (14) is physically associated in transform space with the quantum number for the angular momentum flowing through the image-propagator on S_n which the boundary part signifies in the same way as N in (9) is physically associated with the quantum number for the angular momentum flowing through the propagator $D_s^n(\eta, \eta')$ on C_n which the fundamental part of $D_c^{(n)}(\eta, \eta')$ signifies. The cut-off scale in transform space which N_0 signifies is evidently an increasing function of the geodesic distance a_B.

The expansions (9) and (14) make it evident that the reduction of the eigenvalue problem on C_n to an eigenvalue problem on S_n through the method of images allows for the exploitation of the spherical formulation in the context of any perturbative calculation on C_n although the propagator $D_c^{(n)}(\eta, \eta')$ on the latter no longer admits a direct expansion of the form (9). In order to exemplify the merit of this approach as a preamble to perturbative renormalisation on C_4 the "tadpole" diagram will be evaluated in what follows.

III. The massless tadpole and dimensional renormalisation

The simplest diagrams in the perturbative expansion of the effective action are those representing contributions which depend on $D_c^{(n)}(0)$ ("bubbles" and "tadpoles"). They are known to vanish in all maximally symmetric space-times provided that dimensional regularisation is used [11]. That regulating technique manifests all divergences as poles

at the limit of the relevant space-time dimensionality n through an analytical extension of the latter in the complex plane. The vanishing effect in its context is due to a peculiar cancellation between the ultra-violet and the infra-red divergence in any massless scalar theory. In the present case, the method of images allows for the treatment of both fundamental and boundary part in $D_c^{(n)}(\eta,\eta')$ as propagation on the entire S_n thereby calling naturally into question the persistence of the stated vanishing effect on C_n.

The loop integral in configuration space for any diagram representing $D_c^{(n)}(0)$-related contributions is

$$I(n) = \int_C d^n\eta D_c^{(n)}(\eta,\eta) \tag{15}$$

with integration over C_n. Its fundamental-part related term

$$\int_C d^n\eta \frac{1}{|\eta-\eta'|^{n-2}} \tag{16}$$

is formally divergent at $n=4$ when the $\eta \to \eta'$ limit is considered in the integrand. The evaluation of this term, however, necessitates $n<2$ from the outset. A subsequent analytical extension causes this integral to vanish when continued back to $n=4$. Nevertheless, the boundary-related term of the same loop integral

$$\int_C d^n\eta \frac{1}{|\frac{a_{\eta'}}{a_B}\eta - \frac{a_B}{a_{\eta'}}\eta'|^{n-2}} \tag{17}$$

is finite and non-vanishing at $\eta \to \eta'$ when continued back to $n=4$. As a result, all $D_c^{(n)}(0)$-related contributions to the effective action, are no longer vanishing. The underlying source of these contributions can be traced to the cut-off separation characterising the boundary part of $D_c^n(\eta,\eta')$ in (17). Specifically, the following relation between spherical harmonics defined on a Euclidean n-sphere of radius a and Gegenbauer polynomials

$$\sum_{\alpha=0}^{N} Y_\alpha^N(\eta)Y_\alpha^N(\eta') = F(N,n)C_N^{\frac{n-1}{2}}(z) \tag{18}$$

with

$$z = \frac{\eta.\eta'}{a^2};\ \ F(N,n) = \frac{2N+n-1}{4a^n\pi^{\frac{n}{2}}}\Gamma(\frac{n-1}{2}) \tag{19}$$

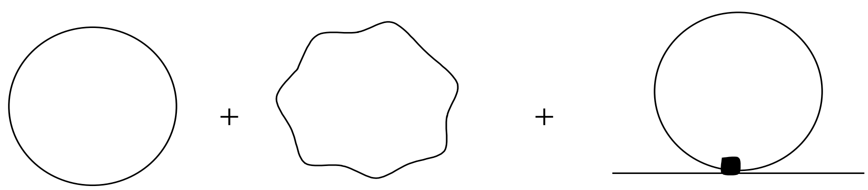
reduces the summation over the “azimouthal” index α

$$\sum_{\alpha=0}^{N} Y_\alpha^N(\frac{a_{\eta'}}{a_B}\eta)Y_\alpha^N(\frac{a_B}{a_{\eta'}}\eta') = \sum_{\alpha=0}^{N} Y_\alpha^N(\eta)Y_\alpha^N(\eta') \tag{20}$$

since

$$\frac{a_{\eta'}}{a_B}\eta.\frac{a_B}{a_{\eta'}}\eta' = \eta.\eta'$$

In effect, the expansion (14) for the boundary part of $D_c^n(\eta,\eta')$ reduces to that in (9) with the stated range of summation over N between 0 and N_0. This reduction allows for manipulations of the boundary part of $D_c^n(0)$ in transform space similar to those of $D_s^n(0)$. It becomes evident, nevertheless, that whereas on S_n the massless propagator vanishes because the $N=0$ term - which in de Sitter space is associated with the infrared divergence - cancels identically against the ultra-violet sum-total of the remaining infinite number of terms [11] the image propagation on S_n expressed by (14) entails only a finite number of terms in the relevant summation over N. Consequently, the same exact cancellation is no longer possible on C_n. The result, albeit finite to zero (bubble) and first order (tadpole) in the renormalised self-coupling λ_R, is of crucial importance to renormalisation as radiative effects at higher orders will generate ultraviolet divergences for the tadpole diagram. In turn, the presence of ∂C may significantly affect the renormalisation program of conformal scalar fields on a Riemannian manifold of constant curvature with the potential for generation of mass and conformally non-invariant counterterms to orders at which the latter have been shown to be absent on S_n. In this respect, it is imperative to explore the potential contributions of all $D_c^n(0)$-entailing diagrams to the effective action



(a) Scalar loop

(b) Graviton loop

(c) Scalar tadpole with bare self-coupling

FIGURE 1. $D_c^n(0)$-related contributions to the effective action

The "bubble" diagrams in fig.(1a) and fig.(1b) not entailing any self-couplings are finite in the context of dimensional regularisation. However, in general power counting terms, they account diagramatically for the simultaneous one-loop contributions to volume and boundary effective Einstein-Hilbert action on any manifold with boundary. Such simultaneous contributions at one loop-level in the absence of self-couplings have been assessed in the general case of non-minimal matter through heat kernel techniques [10]. The Einstein-Hilbert action is, in the case of minimal coupling of matter to gravity, the semi-classical gravitational component of the effective action for a scalar field coupled to the background geometry of C_n and at $n=4$ it assumes the familiar form

$$S_{EH} = -\frac{1}{16\pi G}\left[\int_C d\sigma(R-2\Lambda) + 2\int_{\partial C} d^3x\sqrt{h}K\right] \tag{21}$$

with $d\sigma = a^n d\Omega_{n+1}$ being the element of surface area of the n-sphere embedded in $n+1$ dimensions and with the three-dimensional boundary hypersurface on which the induced metric is h_{ij} being characterised by an extrinsic curvature $K_{ij} = \frac{1}{2}(\nabla_i n_j + \nabla_j n_i)$ the trace of which is $K = h^{ij}K_{ij}$. The surface term in (21) has been the result of distinct approaches to field theoretical formulations on a manifold with boundary [15]. In a variational context it was introduced by Gibbons and Hawking [16] in order to offset the variations of the normal derivatives of the metric on the boundary stemming from the variation of the volume term of the classical action in the context of a fixed metric on the boundary.

The unique conformally invariant bare scalar action on a n-dimensional Riemannian manifold of constant curvature a is [12]

$$S[\Phi_0] = \int_C d\sigma[\frac{1}{2}\frac{1}{2a^2}\Phi_0(L^2 - \frac{1}{2}n(n-2))\Phi_0 - \frac{\lambda_0}{\Gamma(p+1)}\Phi_0^p] \tag{22}$$

provided that $p = \frac{2n}{n-2}, n > 2$. $L_{\mu\nu}$ is the generator of rotations

$$L_{\mu\nu} = \eta_\mu \frac{\partial}{\partial \eta_\nu} - \eta_\nu \frac{\partial}{\partial \eta_\mu} \tag{23}$$

on the embedded sphere. In the physically relevant case of $n = 4$ the self-coupling is that of Φ^4. C_n is characterised by spherical $(n-1)$-dimensional sections of constant Euclidean time τ. In addition to (22) the presence of a boundary on the Riemannian manifold of constant curvature necessitates at the classical level a term of the form $\int_{\partial C} K\Phi^2$. In the same variational context which, in the case of minimal coupling, gives rise to the Gibbons-Hawking boundary term in (21) this term is necessary in order to eliminate the non-vanishing variations of the normal derivative on the boundary stemming from the non-minimal coupling of the conformal scalar field to the background metric. It should be stressed, in addition, that this term which replaces the surface term in (21) is present in the classical action despite the Dirichlet condition of $\Phi(\eta) = 0$ on the boundary since the boundary integral, essentially taken over the product of two scalar fields, is predicated on the coincidence limit $\eta \to \eta'$ which independently results in a divergence in both terms of the propagator in (7) [10]. This term is, in principle, also expected in the effective action at higher loop-orders despite the Dirichlet condition since for any diagram the coincidence limit of integration points is the potential source of divergences on the manifold's volume as well as boundary. The issue which naturally arises in such a context is whether the effective action, obtained by integrating out the interacting quantum scalar field Φ reproduces at higher loop-orders the semi-classical boundary term in question.

The "tadpole" diagram in fig.(1c) formally involving the bare self-coupling λ_0 as well as bare scalar field Φ_0 has the potential for contributions to both the volume scalar effective action as well as to its boundary counterpart of the form $\int K\Phi^2$. Being a diagramatic representation of the first non-trivial correction to the two-point function it yields a contribution to the propagator of the form

$$\frac{1}{2}(-\lambda_0)\int_C d^n\eta D_c^{(n)}(\eta_1,\eta)D_c^{(n)}(\eta,\eta)D_c^{(n)}(\eta,\eta_2) \tag{24}$$

On the grounds of the preceding analysis the loop integral expressing the proper version of the same diagram (no external propagators attached to the loop) is given by (15), (7) and (14) and by virtue of (20) and of the absence of the fundamental-related part it assumes the form

$$\int_C d^n\eta D_c^{(n)}(\eta,\eta) = -\sum_{N=0}^{N_0}\sum_{\alpha=0}^{N}\frac{1}{\lambda_N}Y_\alpha^N(\eta)Y_\alpha^N(\eta)\int_C d^n\eta \tag{25}$$

which, in the context of (18), (19) and (13), yields [11]

$$\int_C d^n\eta D_c^{(n)}(\eta,\eta) =$$

$$-\frac{2N+n-1}{4\pi^{\frac{n}{2}}}\Gamma(\frac{n-1}{2})C_N^{\frac{n-1}{2}}(1)a^{2-n}\sum_{N=0}^{N_0}\frac{(2N+n-1)\Gamma(N+n-1)}{(N+\frac{n}{2})(N+\frac{n}{2}-1)\Gamma(N+1)}\int_C d^n\eta \tag{26}$$

The sum over the angular momentum quantum number N associated with image propagation in (26) amounts to a finite multiplicative factor. In fact, for sufficiently small geodesic distances a_B between the ∂C_n and the n-cap's pole only the first few terms in this sum are relevant.

The volume integral featured in (26) has a well defined value on the Euclidean embedded n-sphere S_n of radius a, namely [17]

$$\int_S d^n\eta = a^n\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \tag{27}$$

On C_n, however, the final integration over the angle θ_n extends from 0 to the boundary-defining value of $\theta_n^{(0)} < \frac{\pi}{2}$. Consequently,

$$\int_C d^n\eta = a^n\int_0^{2\pi}d\theta_1\int_0^{\pi}d\theta_2 sin\theta_2\int_0^{\pi}d\theta_3 sin^2\theta_3 ...\int_0^{\theta_n^0}d\theta_n sin^{n-1}\theta_n$$

$$= a^n\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^{\theta_n^0}d\theta_n sin^{n-1}\theta_n \tag{28}$$

At the limit of space-time dimensionality $n=4$ this is

$$\int_C d^4\eta = a^4\frac{2\pi^2}{\Gamma(2)}[-\frac{1}{3}sin^2\theta_4^0 cos\theta_4^0 - \frac{2}{3}cos\theta_4^0 + \frac{2}{3}] \tag{29}$$

This is the volume integral of C_4 the metric of which manifold is

(30) $$ds^2 = a^2[d\theta_4^2 + sin^2\theta_4 d\Omega_3^2]$$

with $d\Omega_3^2$ being the metric of three spheres characterised by an extrinsic curvature the trace of which is

(31) $$k = \frac{1}{3}\frac{1}{a}cot\theta_4$$

The boundary hypersurface of C_4 is the three sphere of radius $asin\theta_4^0$ and of extrinsic curvature the trace of which is

(32) $$K = \frac{1}{3}\frac{1}{a}cot\theta_4^0$$

with

(33) $$K' \equiv \frac{dK}{d\theta}_{|\partial C} = -\frac{1}{3}\frac{1}{a}\frac{1}{sin^2\theta_4^0}$$

Together (29), (32) and (33) reduce (26) at $n = 4$ to

(34) $$\int_C d^4\eta D_c^{(4)}(\eta, \eta) = \int_{\partial C} d^3\eta K - 6a\int_{\partial C} d^3\eta K K' - \frac{2}{3}\frac{1}{a^2}\pi^{-\frac{1}{2}}\Gamma(\frac{5}{2})\int_{S_4} d^4\eta$$

allowing for a multiplicative factor in each term which involves the sum in (26) and whose significance will be discussed in what follows.

It becomes evident through (24) and (34), for that matter, that, although the "tadpole" diagram entails no primitive divergences in its structure it contributes simultaneously, at any order in the perturbative expansion of λ_0, to both the volume bare action for Φ_0 and to its boundary counterpart. Allowing for inessential multiplicative factors the form of these contributions is

(35) $$(-\lambda_0)\int_{\partial C} d^3\eta K\Phi^2 - 2a(-\lambda_0)\int_{\partial C} d^3\eta K K'\Phi^2 - (-\lambda_0)\int_{C-\partial C} d^4\eta R\Phi_0^2$$

where in the last term use has been made of the Euclidean de Sitter space relation [12]

(36) $$R = \frac{n(n-1)}{a^2}$$

and the fact that the third term in (34) amounts in its entirety to the volume integral over C_4 excluding its boundary. On account of the stipulated Dirichlet condition the radiative contributions in (35) feature the semi-classical scalar field Φ on ∂C as opposed to the bare Φ_0.

As stated, physical requirements enforce the first term in (35) in the clasical action and semi-classical level of the quantum effective action. For that matter, it is also expected at higher orders. However, at the coincidence limit of $\eta \to \eta'$ on the boundary it is $a_{\eta'} = a_B$ and, as a consequence, the boundary-related term in (17) of the loop integral

in (15) which is finite and non-vanishing in the volume reduces to the same form as that of the fundamental-part related term in (16). As a result, the expansion in the aforementioned multiplicative factor stemming from (26) in the first two terms of (35) is no longer truncated at N_0. The peculiar cancellation between the ultra-violet and the infra-red divergence of the massless "tadpole" in dimensional regularisation and the concomitant vanishing effect is again "at work" on the boundary forcing the first two terms in (35) to vanish. As a consequence, the "tadpole's" contribution to the bare scalar action on C_4 and to any order in the perturbative expansion of the bare self-coupling reduces to only the third term in (35). This result is consistent at one-loop order with the absence of matter-related surface counterterms in the effective action on a general manifold with boundary [7].

Allowing for the stated vanishing effect it is, nevertheless, worth noting that the dimensionally consistent form of the radiatively induced terms in (35) as well as the general form of the volume integral in (29) which characterises all Feynman diagrams on C_4 suggest in the context of (32) and (33) that radiative contributions to the two-point function tend to reproduce the form of the classical boundary term in the quantum effective action already at low orders in perturbation. These radiative contributions at $n \to 4$ stem exclusively from the bare Φ_0 and bare self-coupling λ_0 which respectively admit an expansion in terms of Φ and λ [18] as

$$\Phi_0 = Z^{\frac{1}{2}}\Phi \ ; \ Z = 1 + \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{c_{ki}\lambda^i}{(4-n)^k} \tag{37}$$

$$\lambda_0 = \mu^{4-n}[\lambda + \sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda)}{(4-n)^\nu}] \tag{38}$$

and allow, in the presence of self-couplings, for the generalisation of the statement in [10] to the effect that both volume and boundary-related ultra-violet divergences are simultaneously removed. In the absence of self-couplings such a simultaneous process is realised through the renormalisation of the gravitational coupling G in the volume sector of (21). In the present case, in addition to the same renormalisation of G, any counterterm to the "tadpole" diagram generated by overlapping divergences through (37) and (38) at any fixed order in λ contributes, in principle, simultaneously to all three sectors in (35). Disregarding the vanishing effect of the boundary terms in the "tadpole" case such simultaneous contributions appear to be relevant to any diagramatic structure at any higher-loop order in the perturbative expansion of the two-point function. As a result of such perturbative redefinitions at higher orders the bare action on C_n can be seen to feature the tree terms in (35) in addition to the bare action on S_4 expressed by (22) at $n = 4$

$$\begin{aligned} S[\Phi_0] = (-\lambda_0)\int_{\partial C} d^3\eta K\Phi^2 - 2a(-\lambda_0)\int_{\partial C} d^3\eta KK'\Phi^2 + \\ \int_C d\sigma[\frac{1}{2}\frac{1}{2a^2}\Phi_0(L^2 - \frac{1}{2}n(n-2))\Phi_0 + \kappa_0 R\Phi_0^2 - \frac{\lambda_0}{4!}\Phi_0^4] \end{aligned} \tag{39}$$

with κ_0 being a bare non-minimal coupling which vanishes at the semi-classical level. The effective action will necessarily have the same form as the bare action if the bare quantities are replaced by their renormalised counterparts as a result of the summation over vacuum diagrams plus counterterms to any specific order [19].

Although the exact nature of vacuum contributions to the effective action necessitates a detailed renormalisation program to any specific order a qualitative assessment can be elicited from (39). As announced, radiative effects at higher loop orders will necessarily generate the boundary term $\int K\Phi^2$ although the latter is irrelevant to the diagramatic structure of the "tadpole" in fig.(1c). The additional $a\int KK'\Phi^2$ term in the effective action is a purely quantum effect. A consistent variational procedure of the classical Einstein-Hilbert action does not necessitate it [10]. The present calculation suggests at higher orders the simultaneous emergence of the stated two terms on the boundary ∂C_4 and the volume contributions represented by the $\int R\Phi_0^2$ term in (35) and (39). This term is the unique contribution of vacuum effects to order one in loop and bare self-coupling expansion and deserves attention in its own merit. Following an argument in [11] the quadratic expression involving the spherical d' Alembertian $L^2 - \frac{1}{2}n(n-2)$ in (39) relates to wave-function renormalisation effected by contributions to the two-point function represented by diagrams of, at least, two self-coupling vertices. These two vertices are rotationally related through the operator given by (23). The "tadpole" diagram featuring one vertex does not fall in that category and that is reflected in the absence of that operator in its volume contribution. The $R\Phi_0^2$ term in (39) is, for that matter, distinct from the quadratic first term in the same expression. Since that quadratic term is the spherically formulated sum of the kinetic $\phi_0\partial^2\phi_0$ and conformal $\frac{1}{6}R\phi_0^2$ sector for a conformal scalar field in the case of a general manifold the presence of the third volume term in the effective action does not reproduce perturbatively the classical conformal coupling of the scalar field to the background geometry of C_4. This situation contrasts sharply with the absence of conformally non-invariant counterterms for the same massless scalar theory on S_4 [11]. Since this radiative effect arises from the "tadpole" diagram which entails no primitive divergences in dimensional regularisation the lowest order in perturbation at which the generation of this conformally non-invariant counterterm in the effective action is expected is order two in the bare self-coupling expansion. This expectation is predicated on the first non-trivial ultra-violet divergence stemming from the correction at that order to the four-point function for the same scalar theory defined on S_4 [11]. Since that volume divergence is a cut-off scale effect arising from the coincidence of integration points in configuration space it is expected to be topology-independent and persist at least as a leading divergence in the volume of C_4. Consequently, to order two in the expansion of λ_0 the tadpole structure will represent a divergent counterterm to the two point function. Conformal invariance is broken as a result of boundary-related contributions to the effective action. The conformal anomaly for the one-loop effective action has been evaluated through use of heat kernel techniques [4]. The explicit calculation of the conformal anomaly in the context of the above considerations as well as the additional suggestive aspect of this calculation relating to the generation of the two boundary terms in the effective action and to the

simultaneous renormalisation of boundary and volume ultra-violet divergences will be explicitly confirmed through higher-loop renormalisation.

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